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Ultrametric Broken Replica Symmetry RaMOSt

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We propose an ultrametric breaking of replica symmetry for diluted spin glasses in the framework of Random Multi-Overlap Structures (RaMOSt). Our approach permits to bound the free energy through a trial function that depends on a set of numbers over which one has to take the infimum. Such trial function is a first (ultrametric and factorized) example of a bound in the intersection of the probability spaces of the iterative and the RaMOSt theories, and it shows that a "direct dilution" of the Parisi Ansatz is not always exact.

KEY WORDS: diluted spin glasses, replica symmetry breaking, ultrametric overlap structures.

1. INTRODUCTION

In the case of non-diluted spin glasses, M. Aizenman R. Sims and S. L. Starr[(1)] introduced the idea of Random Overlap Structure (ROSt) to express in a very elegant manner the free energy of the model as an infimum over a rich probability space, to exhibit an optimal structure (the so-called Boltzmann one), to write down a general trial function through which one can formulate various ansatz's for the free energy of the model. It was also described how to formulate in particular the Parisi ansatz within this formalism.

In the context of diluted spin glasses, M. Mezard and G. Parisi[(2)] showed how to implement the Replica Symmetry Breaking theory by translating it into the iterative approach. The result is that the Broken Replica Symmetry trial function depends on a nested chain (of Parisi type) of probabilty distributions and the order parameter is always a function. The rigorous proof that the Replica Symmetry Breaking in the sense of ref. 2 yields bounds for the free energy has been given by S. Franz and M. Leone,[(3)] and D. Panchenko and M. Talagrand.[(4)]

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In ref. 5 we extended the concept of ROSt to the one of Random Multi-Overlap Structure (RaMOSt), to deal with diluted spin glasses. Like in the non-diluted case, we could express the free energy of the model by means of the Extended Variational Principle, exhibit the optimal Boltzmann RaMOSt, write down the generic trial function, find a factorization property of the optimal structures. Here we extend in a natural way the Parisi ROSt to an ultrametric RaMOSt. This is interesting to do to check whether a "minimal dilution" of the Parisi theory can be valid. It turns out that such a minimal dilution leads to a trial function that is properly factorized and exact in some regions, not exact in some others (while in some of these regions the iterative method yields exact results). This means that dilute models are deeply different from their infinite connectivity limit, but it is not clear how.

The physics implied by the RaMOSt theory is different (simpler) from the one implied by the iterative method, as it is determined explicitly and entirely by the multi-overlaps and the trial functions depend only on a set of numbers (fixed trial multi-overlaps) and not functions (like the distribution of the primary fields of the iterative approach). The ultrametric trial function we suggest, is the restriction of the bound proved in refs. 3 and 4 to the simpler framework of RaMOSt, but our approach has nothing to share with the iterative method. Nonetheless, our approach allows us to recover the formulas and bounds from the iterative approach in a different and very simple way, provided one leaves certain random variables to be generic instead of making the restriction that we will use to impose ultrametricity in a multi-overlap structure (this connection between the two methods will be clearer later on). One way to look at the two approaches is the following. One has to introduce some variables to infimize over on which the trial free energy depends on. The iterative method considers the cavity fields generic random variables and let their distribution vary. The RaMOSt theory let the chosen trial multi-overlaps and their (probability) weights vary. In the iterative approach the weights are of a given form that cannot change. In the RaMOSt approach the cavity fields obey a certain constraint. We will point out some advantages of the RaMOSt theory, some weak points, some open problems.

We start our treatment by illustrating the Replica Symmetric bound (in Sec. 4), in a very simple way, close to the strategy typically used for non-diluted systems. In this simpler setting we can easily introduce all the ideas we need in the general scheme that we report in detail in Sec. 5. The physical ideas at the basis of our bound are suggested by a particular interpretation of the Parisi theory for the (non-dilute) SK model which we describe in the last Appendices.

2. MODEL, NOTATIONS, DEFINITIONS

Notations:

α, β are non-negative real numbers (degree of connectivity and inverse temperature respectively);

P_ζ is a Poisson random variable of mean ζ;
$\{i_\nu\}, \{j_\nu\}$ are independent identically distributed random variables, uniformly distributed over points $\{1, \ldots, N\}$;
$\{J_\nu\}$, J are independent identically distributed random variables, with symmetric distribution;
$\{\tilde{J}_\nu\}$ are independent identically distributed random variables, with symmetric distribution (different from that of J);
$\mathcal{J}$ is the set of all the quenched random variables above;
$\sigma : i \to \sigma_i$ is a spin configuration;
$\pi_\zeta(\cdot)$ is the Poisson measure of mean ζ;
$\mathbb{E}$ is an average over all (or some of) the quenched variables;
$\omega_\mathcal{J}$ is the Bolztmann-Gibbs average explicitly written below;
Ω_N is a product of the needed number of independent identical copies (replicas) of $\omega_\mathcal{J}$;
$\langle\cdot\rangle$ will indicate the composition of an $\mathbb{E}$-type average over some quenched variables and some sort of Boltzmann-Gibbs average over the spin variables, that will be clear from the context.

We will often drop the dependance on some variables or indices or slightly change notations to lighten the expressions, when there is no ambiguity.

We will consider only the case of zero external field, and hence the Hamiltonian of the system of N sites is, by definition

$$H_N^{VB}(\sigma, \alpha; \mathcal{J}) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu} \sigma_{j_\nu}$$

We follow the usual basic definitions and notations of thermodynamics for the partition function and the free energy per site

$$Z_N\big(H_N^{VB}; \beta, \alpha; \mathcal{J}\big) = \sum_{\{\sigma\}} \exp\big(-\beta H_N^{VB}(\sigma, \alpha; \mathcal{J})\big),$$

$$-\beta f_N(\beta, \alpha) = \frac{1}{N}\mathbb{E} \ln Z_N(\beta, \alpha; \mathcal{J})$$

and $f = \lim_N f_N$.
The Boltzmann-Gibbs average of an observable $\mathcal{O}$ is

$$\omega_\mathcal{J}(\mathcal{O}) = Z_N(\beta, \alpha; \mathcal{J})^{-1} \sum_{\{\sigma\}} \mathcal{O}(\sigma) \exp\big(-\beta H_N^{VB}(\sigma, \alpha; \mathcal{J})\big)$$

The multi-overlaps are defined (using replicas) by

$$q_n = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)} = q_{1\cdots n}$$

Definition 1. *A* ***Random Multi-Overlap Structure*** $\mathcal{R}$ *is a triple* $(\Sigma, \{\tilde{q}_{2n}\}, \xi)$ *where*

- Σ *is a discrete space;*
- $\xi : \Sigma \to \mathbb{R}_+$ *is a system of random weights;*
- $\tilde{q}_{2n} : \Sigma^{2n} \to [0, 1], n \in \mathbb{N}, |\tilde{q}| \leq 1$ *is a positive definite* Multi-Overlap Kernel *(equal to 1 only on the diagonal of* Σ^{2n}*).*

Notice that the RaMOSt just defined is the *minimal* extension of the concept of ROSt to a case where all even multi-overlaps must be considered. This is quite the case when dealing with diluted spin glasses, as a consequence of the fact that here the distribution of the coupling is generic and hence determined by all its moments, while in the non-diluted case the couplings are centered Gaussians and thus determined by the second moment only. That is why all the calculations that in the SK case end up in a single term with the 2-overlaps are replaced here by series with all (even) multi-overlaps. So the SK case can be seen as the one where the series stops at the first term (equivalently, as the infinite connectivity limit) and hence we have a recipe to translate from infinite to finite connectivity and vice versa (diluting), modulo a proper temperature rescaling.

3. PREVIOUS RESULTS

Notice that,[5] with $H_N^{VB} = H$

$$\frac{d}{d\alpha}\frac{1}{N}\mathbb{E}\ln\sum_{\gamma}\xi_\gamma \exp(-\beta H) = \sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)\big(1 - \langle q_{2n}^2\rangle\big). \tag{1}$$

Consider two random variables $\tilde{H}_\cdot(\gamma, \alpha; \tilde{J})$ and $\hat{H}(\gamma, \alpha; \hat{J})$ such that

$$\frac{d}{d\alpha}\mathbb{E}\ln\sum_{\gamma}\xi_\gamma \exp(-\beta \tilde{H}_\cdot) = 2\sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)(1 - \langle \tilde{q}_{2n}\rangle) \tag{2}$$

$$\frac{d}{d\alpha}\frac{1}{N}\mathbb{E}\ln\sum_{\gamma}\xi_\gamma \exp(-\beta \hat{H}) = \sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)\big(1 - \langle \tilde{q}_{2n}^2\rangle\big) \tag{3}$$

and the trial function

$$G_N(\mathcal{R}) = \frac{1}{N}\mathbb{E}\ln\frac{\sum_{\sigma,\tau}\xi_\tau \exp\big(-\beta\sum_{i=1}^{N}\tilde{H}_i\sigma_i\big)}{\sum_\tau \xi_\tau \exp(-\beta\hat{H})}$$

where $\tilde{H}_i$ are independent copies of $\tilde{H}_.$. Then in ref. 5 we proved the following

Theorem 1. (Generalized Bound)

$$-\beta f \leq \lim_{N\to\infty} \inf_{\mathcal{R}} G_N(\mathcal{R});$$

Theorem 2. (Extended Variational Principle)

$$-\beta f = \lim_{N\to\infty} \inf_{\mathcal{R}} G_N(\mathcal{R});$$

Theorem 3. (Factorization of optimal RaMOSt's) *In the whole region where the parameters are uniquely defined, the following Cesàro limit is linear in N and $\bar{\alpha}$*

$$\mathbf{C}\lim_M \mathbb{E}\ln\Omega_M\left\{\sum_\sigma \exp[-\beta(\tilde{H}(\alpha)+\hat{H}(\bar{\alpha}/N))]\right\} = N(-\beta f+\alpha A)+\bar{\alpha}A,$$

where

$$A=\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)(1-\langle q_{2n}^2\rangle),\ \ \tilde{H}=\sum_{i=1}^{N}\tilde{H}_i\sigma_i$$

We will see in the next sections what are the conditions on $\tilde{H}_.$ and $\hat{H}$ in order to obey (2)–(3) when they have a form similar to the Viana-Bray Hamiltonian. Concretely, it is like in the non-diluted case, where the two variables are always the same, just realized in different spaces. In other words, what really changes is Σ, and $\tilde{H}_.$ and $\hat{H}$ assume different representations accordingly.

We want now to construct a trial function with some features. We want it to satisfy the invariance property of the optimal structures, we want it to be some dilution of the Parisi trial function for the SK model and to implement ultrametric breaking of replica symmetry, we want it to depend on the distribution of the original couplings only, as physically we do not expect other probability distributions to play any role, we want it to connect the iterative method with the RaMOSt theory, by being a restriction of the general trial function of the iterative approach.

4. THE REPLICA SYMMETRIC RaMOSt

In this section we find the Replica Symmetric trial function within the RaMOSt approach, with no external field.

The choice of the probability space of the Replica Symmetric RaMOSt is trivial, as we do not really need it, just like in the non-diluted case. Still, it will serve as a guide to the next section.

Here is the interpolating Hamiltonian

$$H(t) = -\sum_{\nu=1}^{P_{\alpha t N}} J_\nu \sigma_{i_\nu}\sigma_{j_\nu} - \sum_{\nu=1}^{P_{2(1-t)\alpha N}} \tilde{J}_\nu \sigma_{i_\nu} - \sum_{\nu=1}^{P_{\alpha t N}} \hat{J}_\nu$$

where $\tilde{J}_\cdot$ and $\hat{J}_\cdot$ are symmetric random variables which might have different distribution (and different from the one of $J_\cdot$). The partition function $Z(t)$ associated to this Hamiltonian is defined in the usual way and the usual derivative yields the following standard calculation (see e.g.[5])

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\mathbb{E}\ln Z_N(t) &= \alpha\mathbb{E}\ln\cosh(\beta J) - 2\alpha\mathbb{E}\ln\cosh(\beta\tilde{J}) + \alpha\mathbb{E}\ln\cosh(\beta\hat{J})\\ &\quad - \alpha\sum_{n>0}\frac{1}{2n}\langle\tanh^{2n}(\beta J)q_{2n}^2 - 2q_{2n}\tanh^{2n}(\beta\tilde{J})\\ &\quad + \tanh^{2n}(\beta\hat{J})\rangle_t.\end{aligned}$$

where the term in $\hat{J}$ is clearly vanishing (if $\hat{J}$ is symmetric) but we put it there because we wanted to add and subtract a certain quantity written in two different ways trying to "compose a square." Expressing the exponential of the part in $\hat{J}$ in terms of hyperbolic cosine and tangent (as opposed to just cancel it trivially with the logarithm) complicates things but yields the right expressions for the quantity to be added and subtracted. The contribution at $t = 0$ to the t-dependent free energy is computed in Appendix A. From the expression above it is clear that the order parameter has to be determined by $\tanh^{2n}(\beta\tilde{J})/\tanh^{2n}(\beta J)$. It is therefore convenient to give such fractions a name by defining the so-called primary field g so that

$$\tanh(\beta\tilde{J}) = \tanh(\beta J)\tanh(\beta g).$$

One can readily check that using this definition the next steps lead to the usual Replica Symmetric trial function which also gives the correct critical point if expanded in power series (at the fourth order). We want instead to perform a specific choice in order to include the Replica Symmetric trial function within the framework of RaMOSt's. Namely let us choose $\tilde{J}$ and $\hat{J}$ such that

$$\tanh(\beta\tilde{J}) = \tanh(\beta J)\tilde{\omega}_{\tilde{\alpha}}(\rho_{k_\nu}),\ \tanh(\beta\hat{J}) = \tanh(\beta J)\tilde{\omega}_{\tilde{\alpha}}(\rho_{k_\nu})\tilde{\omega}_{\tilde{\alpha}}(\rho_{l_\nu}) \qquad (4)$$

where $\tilde{\omega}_{\tilde{\alpha}}(\rho_{k_\nu})$ is the infinite volume limit of the Boltzmann-Gibbs average of a random spin from an auxiliary system with a Viana-Bray one-body interaction Hamiltonian at connectivity $\tilde{\alpha}$. This new system has spins denoted by ρ_k, multi-overlaps denoted by $\tilde{q}_{2n}$, same couplings (independent copies) as the ones of the original system. Notice that given any trial multi-overlap there exists $\tilde{\alpha}$

such that the averaged multi-overlap take that value (see Appendix A). From our choice it is clear that a single $\tilde{\alpha}$ generates a whole sequence $\{\tilde{q}_{2n}(\tilde{\alpha})\}$ of trial multi-overlaps. Our approach is based on the assumption that we can limit our trial functions to such sequences. Notice that the distribution of of $\tilde{J}$ is completely determined by the one of J only, as no other quenched couplings arise.
Now we can use the identities

$$\ln\cosh(\cdot) = \sum_{n=1}^{\infty} \frac{1}{2n} \tanh^{2n}(\cdot),\ \mathbb{E}\ \tilde{\omega}_{\tilde{\alpha}}^{2n}(\rho_{k_\cdot}) = \langle \tilde{q}_{2n} \rangle_{\tilde{\alpha}} = \tilde{q}_{2n}(\tilde{\alpha}) \tag{5}$$

to verify that the terms in $\hat{J}$ actually mutually cancel out and also to get

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\ln Z_N(t) = \alpha \sum_{n>0} \frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)\langle (1-\tilde{q}_{2n}(\tilde{\alpha}))^2 - (q_{2n} - \tilde{q}_{2n}(\tilde{\alpha}))^2 \rangle_t$$

where the t-dependent expectation has definite sign, hence we obtain the Replica Symmetric bound and trial function from the fundamental theorem of calculus and Lemma 1

$$F_{\mathrm{RS}}(\beta, \alpha; \{\tilde{q}_{2n}(\tilde{\alpha})\}) = \ln 2 + \mathbb{E}\ln\cosh\left(\beta \sum_{\nu=1}^{P_{2\alpha}} \tilde{J}_\nu(\tilde{\alpha})\right)$$
$$+ \alpha \sum_{n=1}^{\infty} \frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)(1-\tilde{q}_{2n}(\tilde{\alpha}))^2$$

which is the restriction of the usual Replica Symmetric trial function to the choice (4) and gives the correct annealed solution for $\tilde{\alpha} = 0$, since $\tilde{q}_{2n}(0) = 0$ and $\tilde{J}(0) = 0$. In other words, what we did is to conjecture that we can limit the trial function to those primary fields g with moments (of $\tanh(\beta g)$) satisfying a certain constraint, namely the one given by the multi-overlaps. The reason relies on the Extended Variational Principle of ref. 5 and the results of the next section. After all, on a physical basis we do not expect the occurrence of other probability distributions different from and totally independent of that of the original couplings.

We want now to get the whole trial function in the value at zero of the "interpolating pressure" and we want to be left with a definite sign derivative yielding an immediate bound. This becomes essential in the Replica Symmetry Breaking. The interpolating Hamiltonian is

$$H(t) = -\sum_{\nu=1}^{P_{\alpha t N}} J_\nu \sigma_{i_\nu}\sigma_{j_\nu} - \sum_{\nu=1}^{P_{2(1-t)\alpha N}} \left(\frac{1}{\beta}\ln\frac{\cosh(\beta J)}{\cosh(\beta \tilde{J}_\nu)} + \tilde{J}_\nu \sigma_{i_\nu}\right)$$
$$-\sum_{\nu=1}^{P_{\alpha t N}} \left(\frac{1}{\beta}\ln\frac{\cosh(\beta J)}{\cosh(\beta \hat{J}_\nu)} + \hat{J}_\nu\right) \tag{6}$$

and the generalized trial function is

$$G_N = \frac{1}{N}\mathbb{E}\ln\frac{\sum_\sigma \exp(-\beta\tilde{H}(\sigma))}{\exp(-\beta\hat{H})} = R(0) \tag{7}$$

where

$$\tilde{H}(\sigma) = H(0),\ \hat{H} = H(1) - H_N^{VB}(\sigma),\ R(t) = \frac{1}{N}\mathbb{E}\ln\frac{Z(t)}{\exp(-\beta\hat{H})} \tag{8}$$

and $Z(t) = Z(H(t))$ is defined in the usual way.
Now from Lemma 1 in Appendix A we get

$$G_N = G_{RS}(\tilde{\alpha}) = \ln 2 + \mathbb{E}\ln\cosh\left(\beta\sum_{\nu=1}^{P_{2\alpha}}\tilde{J}_\nu\right) + \alpha\mathbb{E}\ln\cosh(\beta J) - 2\alpha\mathbb{E}\ln\cosh(\beta\tilde{J}) + \alpha\mathbb{E}\ln\cosh(\beta\hat{J})$$

and

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\ln\frac{Z(t)}{\exp(-\beta\hat{H})} = -\alpha\sum_{n>0}\frac{1}{2n}\left\langle \tanh^{2n}(\beta J)q_{2n}^2 - 2q_{2n}\tanh^{2n}(\beta\tilde{J}) + \tanh^{2n}(\beta\hat{J})\right\rangle_t$$

becomes

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\ln\frac{Z(t)}{\exp(-\beta\hat{H})} = -\alpha\sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)\langle(q_{2n} - \tilde{q}_{2n}(\tilde{\alpha}))^2\rangle_t$$

with the usual choices for $\tilde{J}_.$ and $\hat{J}_{..}$.

Since

$$\frac{1}{N}\mathbb{E}\ln\frac{Z(1)}{\exp(-\beta\hat{H})} = -\beta f_N^{VB}$$

the fundamental theorem of calculus and the definite sign of the derivative above provide again the Replica Symmetric trial function and bound that we summarize in the just proved

Theorem 4. *With the choice defined by* (6), (7), *and* (8), $\tilde{H}_.$ *and* $\hat{H}$ *satisfy* (2)–(3), *and*

$$-\beta f(\beta,\alpha) \leq G_{RS}(\tilde{\alpha}) = F_{RS}(\tilde{\alpha}) \quad \forall\,\tilde{\alpha}\ \in [0,\infty].$$

Notice that, if $\beta'^2 = \alpha \mathbb{E} \tanh^2(\beta J)$ is fixed, then

$$\lim_{\alpha \to \infty} F_{RS}^{VB}(\beta, \alpha, \tilde{\alpha}) = F_{RS}^{SK}(\beta')$$

where F_{RS}^{VB} is the Replica Symmetric trial function for the Viana-Bray model constructed in this section, and F_{RS}^{SK} is the Replica Symmetric trial function for SK model.

In the non-dilute case, a complete control of the high temperature regime can be gained by means of the quadratic replica coupling method,[(6)] also when there is an external field. The extension of that method to the Viana-Bray model lead the same result, only when there is no external field.[(7)] An external field reveals an intrinsic pathology of dilute models. But the presence of an external field seems to be pathological for the goodness of the bounds also in the approach we propose in this article, we will comment on this later on. That is why we limit ourselves to the case of zero external field, although mathematically it would be very easy to include it in the treatment.

5. REPLICA SYMMETRY BREAKING AND ULTRAMETRIC RAMOST

In this section we will construct a trial free energy of Parisi type depending on ultrametric trial multi-overlaps. The purpose is to show that the iterative and the RaMOSt theories can be compatible, in the sense that there are trial functions that live both in a RaMOSt and in the probability space of the general trial function of the iterative method. Moreover, we want to show how one can construct a trial function depending on ultrametric multi-overlaps, extending the Parisi ultrametricity to the diluted case. Our main goal thus is not to find the exact value of the free energy, nor to get closer to it than one can get with the iterative trial function. In fact, we will construct a trial function that turns out to be some restriction of the iterative trial function. Hence the trial function in this section cannot be closer to the true free energy than the iterative one.

We need to generalize the ideas of the previous section, in particular the second identity in (5) and the preceding discussion. Given any partition $\{x^a\}_{a=0}^K$ of the interval [0, 1], there exists a sequence $\{\tilde{\alpha}_a\}_{a=0}^K \in [0, \infty]$ such that $\tilde{q}_{2n}(\tilde{\alpha}_a) = x_a - x_{a-1}$. In other words, a sequence $\{\tilde{\alpha}_a\}_{a=0}^K \in [0, \infty]$ generates for each $n \in \mathbb{N}$ a partition of [0, 1] considered as the set of trial values of $\tilde{q}_{2n}$, provided the $\tilde{\alpha}_a$ are not too large

$$\sum_{a \leq K} \tilde{q}_{2n}(\tilde{\alpha}_a) \leq 1. \tag{9}$$

Again, we limit our trial multi-overlaps to belong to partitions generated in this way. This implies that the points of the generated partitions tend to get closer to zero as n increases. This is good, since in any probability space $\langle \tilde{q}_{2n} \rangle$ decreases as n increases and therefore the probability integral distribution functions tend to grow faster near zero.

We can then define $\tilde{W}_\gamma$, for $\gamma \in \mathbb{N}^K$, through

$$\tilde{W}_\gamma(\bar{J}, k_\nu) = \tilde{\omega}_{\tilde{\alpha}_1}(\rho_{k_\nu})\bar{J}_{\gamma_1} + \cdots + \tilde{\omega}_{\tilde{\alpha}_K}(\rho_{k_\nu})\bar{J}_{\gamma_1\cdots\gamma_K}$$

with $\bar{J}_{\cdot} = \pm 1$ independent identically distributed symmetric random variables.

Definition 2.

$$\tilde{q}_{\gamma^1\ldots\gamma^{2n}} = \left(\tilde{q}^1_{2n} - \tilde{q}^0_{2n}\right)\delta_{\gamma^1_1\cdots\gamma^{2n}_1} + \cdots + \left(\tilde{q}^K_{2n} - \tilde{q}^{K-1}_{2n}\right)\delta_{\gamma^1_1\cdots\gamma^{2n}_1}\cdots\delta_{\gamma^1_K\cdots\gamma^{2n}_K}$$

is the ultrametric $2n$*-overlap.*

Clearly this is just a kind of ultrametricity, imposed for each (even) number of replica *individually*.

The choice of $\tilde{W}_\gamma$ imposes an ultrametric structure since

$$\begin{aligned}
\mathbb{E}(\tilde{W}_{\gamma^1}\cdots\tilde{W}_{\gamma^{2n}}) &= \mathbb{E}\tilde{\omega}^{2n}_{\tilde{\alpha}_1}(\rho_{\cdot})\delta_{\gamma^1_1\cdots\gamma^{2n}_1} + \cdots + \mathbb{E}\tilde{\omega}^{2n}_{\tilde{\alpha}_K}(\rho_{\cdot})\delta_{\gamma^1_1\cdots\gamma^{2n}_1}\cdots\delta_{\gamma^1_K\cdots\gamma^{2n}_K} \\
&= \tilde{q}_{2n}(\tilde{\alpha}_1)\delta_{\gamma^1_1\cdots\gamma^{2n}_1} + \cdots + \tilde{q}_{2n}(\tilde{\alpha}_K)\delta_{\gamma^1_1\cdots\gamma^{2n}_1}\cdots\delta_{\gamma^1_K\cdots\gamma^{2n}_K} \\
&= \left(\tilde{q}^1_{2n} - \tilde{q}^0_{2n}\right)\delta_{\gamma^1_1\cdots\gamma^{2n}_1} + \cdots + \left(\tilde{q}^K_{2n} - \tilde{q}^{K-1}_{2n}\right)\delta_{\gamma^1_1\cdots\gamma^{2n}_1}\cdots\delta_{\gamma^1_K\cdots\gamma^{2n}_K} \\
&\equiv \tilde{q}_{2n} = \tilde{q}_{\gamma^1\ldots\gamma^{2n}}.
\end{aligned}$$

If we also define

$$\hat{W}_\gamma = \tilde{W}_\gamma(\bar{J}, k_\nu)\tilde{W}_\gamma(\bar{J}', l_\nu)$$

where $\bar{J}'$ denotes independent copies of $\bar{J}$, we have

$$\mathbb{E}(\hat{W}_{\gamma^1}\cdots\hat{W}_{\gamma^{2n}}) = \tilde{q}^2_{2n} = \tilde{q}^2_{\gamma^1\ldots\gamma^{2n}} = [(\tilde{q}^1_{2n})^2 - (\tilde{q}^0_{2n})^2]$$

$$\delta_{\gamma^1_1\cdots\gamma^{2n}_1} + \cdots + [(\tilde{q}^K_{2n})^2 - (\tilde{q}^{K-1}_{2n})^2]\delta_{\gamma^1_1\cdots\gamma^{2n}_1}\cdots\delta_{\gamma^1_K\cdots\gamma^{2n}_K}$$

where the last expected equality can be easily verified by direct calculation. We clearly have in mind the case $\tilde{q}^0_{2n} = 0$, $\tilde{q}^K_{2n} = 1$ (which implies the equal sign holds in (9)).

Now given a set of weights ξ_γ, $\gamma \in \mathbb{N}^K$, we can state the next

Proposition 1. *There exist $\tilde{H}$, $\hat{H}$ satisfying (2)–(3) with $\tilde{q}$ ultrametric.*

Before proving this proposition, let us consider the usual set of weights $\xi_\gamma(m_1, \ldots, m_K)$, $\gamma = (\gamma_1, \ldots, \gamma_K)$ associated to the Random Probability Cascade of Poisson–Dirichlet Processes through which one can express formulas of Parisi type (see e.g. ref. 4). Then take the trial function

$$G_N = \frac{1}{N}\mathbb{E}\ln\sum_{\gamma,\sigma}\xi_\gamma \exp(-\beta\tilde{H}_\gamma) - \frac{1}{N}\mathbb{E}\ln\sum_{\gamma}\xi_\gamma \exp(-\beta\hat{H}_\gamma).$$

Notice that the trial function above is the usual difference between the "cavity" term and the "internal" term. Denoting by X the map

$$X : \tilde{\alpha}_a \to m_a$$

satisfying (9) we can consider the trial function as a function $G(X)$ of X. We will prove the proposition above together with the next

Theorem 5. *The ultrametric trial function $G(X)$ satisfies the bound*

$$-\beta f(\beta,\alpha) \le \inf_X G(X)$$

as in Theorem 1, it enjoys the factorization property as in Theorem 3 (in the sense that $\tilde{H}$ and $\hat{H}$ are independent, and each spin yields the same independent contribution) and it reduces to the Parisi trial function for the SK model in the infinite connectivity limit.

Proof. Consider the interpolating Hamiltonian

$$H_\gamma(t) = H^{VB}(t) + \tilde{H}_\gamma(1-t) + \hat{H}_\gamma(t)$$

where

$$\tilde{H}_\gamma = -\sum_{\nu=1}^{P_{2\alpha N}}\left(\frac{1}{\beta}\ln\frac{\cosh(\beta J)}{\cosh\left(\beta\tilde{J}_\nu^\gamma\right)} + \tilde{J}_\nu^\gamma\sigma_{i_\nu}\right)$$

$$\hat{H}_\gamma = -\sum_{\nu=1}^{P_{\alpha N}}\left(\frac{1}{\beta}\ln\frac{\cosh\left(\beta J\right)}{\cosh(\beta\hat{J}_\nu^\gamma)} + \hat{J}_\nu^\gamma\right)$$

and t is understood to multiply the connectivity α. Consider

$$R(t) = \frac{1}{N}\mathbb{E}\ln\frac{\sum_{\gamma,\sigma}\xi_\gamma\exp(-\beta H_\gamma(t))}{\sum_\gamma\xi_\gamma\exp(-\beta\hat{H}_\gamma)},\ \ G_N = R(0)$$

This time let us chose $\tilde{J}_\gamma$, $\hat{J}_\gamma$ of the form

$$\tanh(\beta\tilde{J}_\gamma) = \tanh(\beta J)\tilde{W}_\gamma,\ \ \tanh(\beta\hat{J}_\gamma) = \tanh(\beta J)\hat{W}_\gamma$$

and compute the usual t-derivative

$$\frac{d}{dt}R(t) = \alpha\mathbb{E}\sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)\,\mathbb{E}\left[\Omega_t^{2n}(\sigma_{i_\nu}\sigma_{j_\nu})\right.$$
$$\left.-2\Omega_t^{2n}(\tilde{W}_\gamma\sigma_{i_\nu}) + \Omega_t^{2n}(\hat{W}_\gamma)\right]$$

where the Ω_t is the generalized Boltzmann-Gibbs average with the weights $\xi_.$ and the Hamiltonian $H(t)$.
It is obvious that

$$\mathbb{E}\Omega_t^{2n}(\tilde{W}_\gamma \sigma_{i_\nu}) = \langle \tilde{q}_{2n} q_{2n} \rangle_t, \ \mathbb{E}\Omega_t^{2n}(\hat{W}_\gamma) = \langle \tilde{q}_{2n}^2 \rangle_t.$$

This proves the proposition.

The RaMOSt is thus equipped with all the ingredients we need and we finally obtain

$$\frac{d}{dt}R(t) = -\alpha \sum_{n>0} \frac{1}{2n} \mathbb{E} \tanh^{2n}(\beta J) \langle (\tilde{q}_{2n} - q_{2n})^2 \rangle_t \tag{10}$$

which is exactly the same expression as in Eq. (5) of ref. 5 except here the trial multi-overlaps are not the Boltzmann ones, but rather some ultrametric ones, in the strictest analogy with the Parisi ROSt for SK. From (10) we clearly get the ultrametric bound that we wanted to prove.

Notice that $G(X)$ does not depend on N, thanks to the same calculations that led to Lemma 1 in Appendix A. Moreover $\tilde{W}$ and $\hat{W}$ are chosen to be independent, therefore the factorization property of the optimal RaMOSt's illustrated in ref. 5 holds:

$$\mathbb{E} \ln \Omega_\xi [c_1 \cdots c_N \exp(-\beta \hat{H}(\bar{\alpha}/N))] = NB + \bar{\alpha} A$$

for some B, and we used the notation[5]

$$c_1 \cdots c_N = \sum_\sigma \exp(-\beta \tilde{H}).$$

Finally, a simple interpolation between the Parisi trial free energy (B.14), at the K-th level of replica symmetry breaking, and $G(X)$, at the same level of replica symmetry breaking, shows that in the infinite connectivity limit the two quantities coincide (modulo the proper temperature rescaling) since the infinite connectivity kills all the multi-overlaps but the 2-overlap, and the latter is the same ultrametric one in the two trial functions for each model. The trial values of the 2-overlap are the same in both trial functions. If we choose the trial values for the SK model, then this determines the sequence of trial auxiliary connectivities to be used in the VB model. Vice versa, given a sequence of auxiliary connectivities, the dilute trial function reduces to the SK one with the same values of the trial 2-overlap. In more general and abstract terms, the constraints (2)–(3) are such that a RaMOSt reduces to a ROSt with the same overlap kernel in the infinite connectivity limit.

What we did is, in other words, to "dilute" (B.12)–(B.13) as □

$$\mathbb{E}(\tanh(\beta \tilde{J}_{\gamma^1}) \cdots \tanh(\beta \tilde{J}_{\gamma^{2n}})) = \mathbb{E} \tanh^{2n}(\beta J) \tilde{q}_{\gamma^1 \dots \gamma^{2n}},$$

$$\mathbb{E}(\tanh(\beta \hat{J}_{\gamma^1}) \cdots \tanh(\beta \hat{J}_{\gamma^{2n}})) = \mathbb{E} \tanh^{2n}(\beta J) \tilde{q}_{\gamma^1 \dots \gamma^{2n}}^2.$$

Notice that X together with $\tilde{\alpha}_a \to \tilde{q}_{2n}^a - \tilde{q}_{2n}^{a-1}$ induces a map

$$X_{2n}(q) = m_a,\ \tilde{q}_{2n}^{a-1} \leq q < \tilde{q}_{2n}^a.$$

As a side remark, notice that the fundamental theorem of calculus applied to (3) implies that in any RaMOSt the part in $\hat{H}$ of G_N has the usual integral form like in the non-diluted case

$$\alpha \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \tanh^{2n}(\beta J) \int_0^1 q X_{2n}(q)\, dq$$

where X_{2n} includes the integration in $d\alpha$ (see Appendi B). In the particular case of the Boltzmann RaMOSt, this is the "internal energy term" with the Boltzmann distribution of the multi-overlaps, since it has the integral form above even without integrating back in $d\alpha$ (see[5]). In the Ultrametric RaMOSt, the corresponding distribution X_{2n} is not the usual Parisi one that would yield

$$\alpha \sum_{n=1}^{\infty} \frac{1}{2n} \mathbb{E} \tanh^{2n}(\beta J) \left(1 - \sum_{a}^{K} (m_{a+1} - m_a) \left(q_{2n}^a\right)^2 \right) \tag{11}$$

while this is instead the case for the SK model. This means that the physics of the model and the interpretation of the parameters $\{m_a\}$ are still quite obscure. In order to make the internal energy part have the same form as in the Boltzmann RaMOSt, one could consider for instance, among other possibilities, starting from (11), leaving $\hat{H}$ out of the interpolation and then try to deduce the proper choice of $\tilde{H}$.

Another remark. The trial function of the iterative method can be written using the same weights that we used here.[4] Then the cavity fields have generic distributions, which are the parameters to infimize over. The trial function we got could be obtained by imposing a restriction to such parameters in order to make the iterative trial function live in a RaMOSt (with ultrametric multi-overlaps). Hence in our example we make a very special choice both for the distribution of the cavity fields, and for the weights of the RaMOSt. This can easily mean simplifying too much. In fact, when there is an external field and the connectivity is smaller than one, the Replica Symmetric trial function of the iterative method is exact, while the Replica Symmetric trial function of the previous section is not (it is not too difficult to prove it through an expansion if powers of α. One could try and break the Replica Symmetry in order to get the exact value of the free energy, but it would be very unphysical and it would give non-selfaveraging overlaps and hence a strict inequality in (10)). Despite its simplicity, the trial function of this section is enough to provide an example of trial function with many good qualities. It is exact when there is no external field in the high temperature regime, it has the invariance property of the optimal structures, it exhibits ultrametricity, it depends on a few trial values of the auxiliary connectivity through which one can vary all the trial multi-overlaps simultaneously, it depends on a natural (decreasing) sequence of trial multi-overlaps as physically expected, it connects the iterative and the RaMOSt approaches, it does not need to introduce new generic probability distributions for the couplings, it is easier to compute than the iterative one, it gives the Parisi trial function in the infinite connectivity limit, it belongs to a RaMOSt (and for the RaMOSt's the Extended Variational Principle has been proven, this is not the case in the iterative method).

6. CONCLUSIONS

The ROSt approach is physically very deep. One takes an auxiliary system with weights and a trial overlap, then a cavity field and an internal field, both Gaussian like the Hamiltonian. The two Gaussian variables stay the same, what changes is the space where they are defined. The trial overlap determines their covariance, which is the parameter to infimize over. A specific form of the trial function, like the Parisi one, is associated to a specific choice of the ROSt (space and weights). Letting the weights to be generic allows one to prove the extended variational principle. In dilute spin glasses we have the analogous structure (RaMOSt) with extended variational principle, and the iterative trial function, which is of the same form (with the same exponents) as the Parisi trial function for the SK model. Now, there are some natural steps to take. The iterative trial function *must* be somehow reduced to a multi-overlap structure (because of Sec. 3). The trial function must be constructed in a RaMOSt, and it is physically expected that the randomness of the cavity and internal fields be determined by the distribution of the original couplings (like for non-dilute spin glasses). It is interesting to find a minimal extension of the Parisi ROSt and to see whether it gives a good trial function. It is also interesting to see whether it is possible to construct a trial function with ultrametric trial multi-overlaps. Well, a trial function fulfilling all these requirements (with the proper invariance property) has been found, in the previous section, and it is not exact in some regions. We think such result is interesting, though of negative nature, and that the trial function is an instructive starting point for further developments. In any case, some considerations arise. The specific restriction of the iterative cavity fields we performed could be wrong, or else the weights of the RaMOSt might not be the right ones. The latter would be quite surprising (and also imply that iterative method is wrong too). But if the restriction we exhibited is wrong, it means that either ultrametricity does not take place in diluted spin glasses, or at least it must be implemented in a radically new way, as the dilution of the Parisi Ansatz is wrong. It is thus important to understand how the physics of dilute spin glasses is different from the non-dilute ones, and how to restrict the iterative trial function.

APPENDIX A: THE CAVITY ENERGY

Let us compute the one-body interaction free energy.

$$\frac{1}{N}\mathbb{E}\ln\sum_{\sigma}\exp\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\sigma_{i_\nu}\right)=\frac{1}{N}\mathbb{E}\ln\sum_{\sigma}\exp\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\sum_{i=1}^{N}\delta_{i,i_\nu}\sigma_{i_\nu}\right)$$
$$=\ln 2+\frac{1}{N}\mathbb{E}\ln\prod_{i=1}^{N}\cosh\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\delta_{i,i_\nu}\right)$$
$$=\ln 2+\mathbb{E}\ln\cosh\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\delta_{1,i_\nu}\right).$$

In the expression above, k (out of m) of the i_ν's will be equal to 1 with probability

$$\binom{m}{k}\left(\frac{1}{N}\right)^{k}\left(\frac{N-1}{N}\right)^{m-k}$$

and therefore

$$\frac{1}{N}\mathbb{E}\ln Z_N^{(1)}=\ln 2+\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left[e^{-2\alpha(N-1)}e^{-2\alpha}\frac{1}{m!}(2\alpha)^{m-k}(2\alpha)^{k}N^{m}\right.$$
$$\left.\frac{m!}{k!(m-k)!}\frac{1}{N^k}\frac{(N-1)^{m-k}}{N^{m-k}}\,\mathbb{E}\ln\cosh\left(\beta\sum_{\nu=1}^{k}\tilde{J}_{\nu}\right)\right].$$

Now the formula

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}a_{m-k}b_k=\left(\sum_{m=0}^{\infty}a_m\right)\left(\sum_{k=0}^{\infty}b_k\right)$$

applies here and yields the following

Lemma 1.

$$\frac{1}{N}\mathbb{E}\ln\sum_{\sigma}\exp\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\sigma_{i_\nu}\right)=\ln 2+\mathbb{E}\ln\cosh\left(\beta\sum_{\nu=1}^{P_{2\alpha}}\tilde{J}_{\nu}\right)$$

which contains two of the terms of the Replica Symmetric trial functional

Remark: it is easy to see that

$$\frac{d}{d\alpha}\frac{1}{N}\mathbb{E}\ln\sum_{\sigma}\exp\left(\beta\sum_{\nu=1}^{P_{2\alpha N}}\tilde{J}_{\nu}\sigma_{i_\nu}\right)=2\sum_{n>0}\frac{1}{2n}\mathbb{E}\tanh^{2n}(\beta J)(1-\langle q_{2n}\rangle)$$

and since for any $m = 0, 1, \ldots$

$$\frac{d}{d\alpha}\pi_\alpha(m) \to 0 \ \Leftarrow \ \alpha \to \infty$$

where $\pi_\alpha(m)$ is the Poisson measure of mean α, we deduce for all n

$$\langle q_{2n} \rangle \to 1 \ \Leftarrow \ \alpha \to \infty.$$

APPENDIX B: PARISI THEORY OF SK

Let us recall the well known SK Hamiltonian, which is defined as a centered Gaussian with covariance given by an overlap

$$H_N^{(SK)}(J) = -\frac{1}{\sqrt{N}} \sum_{i,j}^{1,N} J_{ij}\sigma_i\sigma_j$$

The cavity field $\tilde{H}_i$ acting on the spin σ_i of the Parisi theory is given by the following decomposition of J

$$-\tilde{H}_i = \tilde{J}_\gamma = \sqrt{\tilde{q}_1} J_i^{\gamma_1} + \cdots + \sqrt{\tilde{q}_K - \tilde{q}_{K-1}} J_i^{\gamma_1 \cdots \gamma_K}, \ i = 1, \ldots, N$$

where the γ indexes are the ones of the Random Probability Cascades of Poisson-Dirichlet processes, used for the weights $\xi_\gamma(m_1, \ldots, m_K)$ to express in a compact way the nested expectations of Parisi formula. The couplings of the cavity field are related to the original ones by the trial overlaps

$$\mathbb{E}(\tilde{J}_\gamma \tilde{J}_{\gamma'}) = \mathbb{E}(J^2)\tilde{q}_{\gamma\gamma'}. \tag{B.12}$$

Each trial overlap $\tilde{q}_a$ from the assumed partition of $[0, 1]$ can be obtained as the overlap in an auxiliary system with a one-body interaction (for simplicity) with couplings J modulated by a suitable strength $\sqrt{x_a}$, thanks to the monotone dependance of the overlap on x, i.e. $\tilde{q}_a = \tilde{q}(x_a)$. But we can also put

$$\tilde{q}_a - \tilde{q}_{a-1} = \tilde{q}(x_a) - \tilde{q}(x_{a-1}) = \tilde{q}^a = \tilde{q}(x^a), \quad \sum_{a=1}^{K} \tilde{q}^a = 1$$

and re-write the cavity field as

$$-\tilde{H}_i = \sqrt{\tilde{q}(x^1)} J_i^{\gamma_1} + \cdots + \sqrt{\tilde{q}(x^K)} J_i^{\gamma_1 \cdots \gamma_K}.$$

The ultrametricity is intrinsic in the $\tilde{H}_i$'s, as can be easily checked by their covariance, which is the only quantity that is related to both the overlap and the generalized bound (see ref.(1)). The internal energy is therefore expressed introducing (see ref. 1)

$$-\hat{H} = \hat{J}_\gamma = \sqrt{\tilde{q}_1^2} J^{\gamma_1} + \cdots + \sqrt{\tilde{q}_K^2 - \tilde{q}_{K-1}^2} J^{\gamma_1 \cdots \gamma_K},$$

or equivalently

$$\mathbb{E}(\hat{J}_\gamma \hat{J}_{\gamma'}) = \mathbb{E}(J^2) \tilde{q}_{\gamma\gamma'}^2, \tag{B.13}$$

and the trial function can be written as

$$G_P = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\gamma,\sigma} \xi_\gamma \exp\Big(- \beta \sum_{i=1}^N \tilde{H}_i \sigma_i \Big)}{\sum_\gamma \xi_\gamma \exp(-\beta \hat{H})} = -\beta f_{K-BRS}(X) \tag{B.14}$$

where X is the Parisi order parameter. Notice that there is a $y^a = y(x^a)$ such that

$$\tilde{q}_a^2 - \tilde{q}_{a-1}^2 = \tilde{q}^2(y^a)$$

and that $\{y^a\}$ is determined by $\{x^a\}$ since

$$\tilde{q}_a^2 - \tilde{q}_{a-1}^2 = (\tilde{q}_a - \tilde{q}_{a-1})(\tilde{q}_a + \tilde{q}_{a-1}) = \tilde{q}(x^a) \left(2 \sum_{r=1}^{a} \tilde{q}(x^r) + \tilde{q}(x^{a+1}) \right)$$

so that the trial function G can be expressed in terms of the x^a only.

Moreover, it is easy to see that

$$\frac{1}{N} \mathbb{E} \ln \sum_\gamma \xi_\gamma \exp(-\beta \hat{H}) = \frac{\beta^2}{2} \int_0^1 q X(q) dq = \frac{\beta^2}{2} \frac{1}{2} (1 - \langle q^2 \rangle)$$

using for instance integration by parts or Fubini theorem. The second equality above holds in full generality, for any average $\langle \cdot \rangle$ in some space of a random variable q between zero and one, the distribution of which can be denoted by X. In particular, X can be the one associated to the Boltzmann-Gibbs measure or the Parisi one: in the former case $\hat{H}$ is given in ref.,(1) the latter case has just been illustrated.

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